https://doi.org/10.31674/book.2023pcbs11

Profiling the Dynamics of a Stochastic Quantum System Modulated by Rapid Time-Periodic Stimulus

Anindita Shit
Department of Chemistry, Kandi Raj College, Kandi, Murshidabad, West Bengal, India

Corresponding Author's Email: anindita.pchem@gmail.com

ABSTRACT

The dynamics of a quantum dissipative system driven by a fast-oscillating high-frequency periodic field has been investigated here in the semi classical regime. As the system can no longer be considered to be in a conventional thermal equilibrium situation, the theoretical study of these non-adiabatically driven systems is cumbersome. Here, the frequency of external modulation is substantially higher as compared to all other pertinent system frequencies. Beginning with a quantum system-reservoir Hamiltonian with explicit time dependence, derive a time-independent effective c-number generalized Langevin equation (GLE) at leading order by exploiting a technique from the protocol of MSPT (multiple scale perturbation theory). Within the so obtained c-number GLE (which does not include explicit time-dependence), the original system dynamics can be evaluated by its slow part with the time-independent effective potential. Here the dynamics of the slow-part are explored perturbatively in terms of ω^{-1} (ω stands for the frequency of time-periodic oscillating force) up to the order ω^{-4}. Modulation of the parameters of the effective potential often provides a potential avenue to increase or abate the escape probability of the system from the region of attraction of the potential well.

Keywords: *Quantum Dissipative Systems; Dynamical Processes; Langevin Equation; Effective Potential; Multiple Scale Perturbation Analysis*

Introduction

The question of how a system responds to an external rapid periodic perturbation is one of the elementary problems in the field of chemical dynamics in condensed phases and has importance in classical as well as quantum mechanics (Bukov, D'Alessio & Polkovnikov, 2015; Hänggi, Luczka & Spiechowicz, 2020; Spiechowicz, Hänggi & Luczka, 2022). It plays primitive roles in many phenomenal applications, including escape from a metastable state (Kapitsa, 1951; Landau & Lifshitz, 2013; Paul, 1990). Classical or quantum time-dependent systems exhibit more difficult behaviour than corresponding time-independent systems. Here, it is claimed that for a clear insight (both qualitative and quantitative) into the dynamics of a rapidly modulated system, multiple scale perturbation theory (MSPT) may be used as an effective tool. Theoretical analysis of nonadiabatically driven systems is complicated, since one may no longer assume that the system is in thermal equilibrium. On the other hand, for equilibrium systems, the exponential part in the escape rate expression can be found as the height of the free energy barrier, but for nonequilibrium systems, the pre-factor case is even more complicated as there are no general relations from which it can be achieved.

Literature Review

There have been many attempts to resolve the nonadiabatic response problem in many contexts (Gammaitoni *et al.,* 1998; Jung, 1993; Denisov *et al.,* 2006; Denisov *et al.,* 2007; Denisov, Polyakov & Lyutyy, 2011; Kim *et al.,* 2010; Spiechowicz *et al.,* 2023). Thus, the development of an elaborate theory of the dynamics of the system under the impact of modulation is indispensable as well as useful. Here the dynamics of a quantum system has been presented which is driven by a rapid, time-periodic, space-dependent oscillating field, one of the most important classes of non-equilibrium systems. It is always hard to anticipate the qualitative properties of systems with periodic modulation as compared to the dynamics of structurally similar time independent systems, which are easy to perceive. Here, the dynamics of time-dependent modulated systems in which there exists a clear separation of time scales will be related to the dynamics of time-independent ones. Thus, by using the experience with the dynamics of time-independent systems, both qualitative and quantitative studies of the dynamics of time-dependent systems can be done. Here the Brownian dynamics is analysed in the quantum regime in terms of an effective time-independent Hamiltonian by invoking systematic expansion of the time-dependent system-bath Hamiltonian in (being the driving frequency) with a systematic time scale separation. It is found that the slow part of the motion may be interpreted by a time independent effective potential and modulation changes the activation barrier, which provides an effective control of the escape rate and a precise measurement of the system parameters (Floquet, 1883; Shirley, 1965; Chen *et al.,* 1973; Shit, Chattopadhyay & Ray Chaudhuri, 2012a; Shit, Chattopadhyay & Ray Chaudhuri, 2012b; Shit, Chattopadhyay & Ray Chaudhuri, 2012c; Shit, Chattopadhyay & Ray Chaudhuri, 2013). For rapidly oscillating field frequencies, where the driving becomes nonadiabatic, the expected major effect due to the field would "modulate" the system by changing its potential. Here the study such driven systems in a very general form for a wide range of driving frequencies, which goes far beyond the adiabatic limit. The description narrated below can be viewed as a generalization of the work of Kapitsa-Landau-Lifshitz (Kapitsa, 1951; Landau & Lifshitz, 2013) within the frame of system-bath model. This work may be implemented to explore the escape dynamics and the trapping mechanism for the Brownian particle moving in a space-dependent rapidly oscillating field (Shit, 2016).

Discussion

Consider the system to be a quantum particle of mass *m* associated with a bath consisting of harmonic oscillators with characteristic frequencies $\{\Omega_j\}$ and masses $\{m_j\}$. The system is evolving under the influence of an external periodic potential $\hat{V}_1(\hat{x}, \omega t)$ [where ω is the frequency of the external modulation]. Note that the average of the time-dependent periodic potential $\hat{V}_1(\hat{x}, \omega t)$ over a period $[\tau = (2\pi/\omega)]$ can be delineated as follows:

$$\hat{V}_1\big(\hat{x}, \omega(t+\tau)\big) = \hat{V}_1(\hat{x}, \omega t);$$

$$\frac{1}{\tau}\int_0^\tau \hat{V}_1(\hat{x}, \omega t) = 0$$

As expected, at t = 0 (when there is no external driving force), the harmonic bath is in thermal equilibrium with the system. Note that at t = 0+, $\hat{V}_1(\hat{x}, \omega t)$ is turned on and the system starts moving in the external force. The total Hamiltonian (Weiss, 2012; Zwanzig, 1961; Zwanzig, 1973) can be constructed by a system part, a bath part and the system-bath interaction part:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}_0(\hat{x}) + \hat{V}_1(\hat{x}, \omega t) + \sum_{j=1}^{N} \left\{ \frac{\hat{p}_j^2}{2m_j} + \frac{1}{2} m_j \Omega_j^2 \left(\hat{x}_j - \frac{c_j \hat{x}}{m_j \Omega_j^2} \right)^2 \right\} \tag{1}$$

Here $\hat{x}$ is the position operator, and $\hat{p}$ corresponds to the momentum operator of the system. $\{\hat{x}_j, \hat{p}_j\}$ stands for coordinate and momentum operators for the harmonic bath and obeys the relations $[\hat{x}_j, \hat{p}_j] = i\hbar\delta_{ij}$. The coupling between system and bath is linear in nature characterized by the coupling parameter c_j, $\hat{V}_0$ stands for the system potential in absence of coupling. Eq. (1) explicitly depicts that each of the harmonic bath oscillator is shifted relative to the system by an amount dependent on their correlative coupling that can be considered as a compensation of a renormalization of the system potential (Weis, 2012). In the present case, a revised bath Hamiltonian controls the proper distribution of initial states which is expressed as:

$$\hat{H}_B = \sum_{j=1}^{N} \left[\frac{\hat{p}_j^2}{2} + \frac{1}{2} m_j \Omega_j^2 \left(\hat{x}_j - \frac{c_j \hat{x}}{m_j \Omega_j^2} \right)^2 \right], \text{ at } t = 0\,. \tag{2}$$

Exploiting Eq. (1), one can design Hamilton's equation of motion (EOM) for both the system variables as well as the bath degrees of freedom. These are two differential equations due to the mutual interplay between the system and the bath. The Langevin equation (which consists of both dissipation and fluctuation terms) can be obtained by solving the required expressions of the bath degrees of freedom and subsequently exploiting these equations into the corresponding equations of the system variables. The statistical properties of the system degrees of freedom are characterized by the distribution of the initial conditions of the bath degrees of freedom via the fluctuation-dissipation relation. Here, the microscopic structure of the dissipative term and the fluctuating force comprise the initial conditions of bath degrees of freedom. The operator Langevin equation to describe the evolution of the system can be expressed as:

$$\dot{\hat{x}} = \frac{\hat{p}}{m}$$

$$\dot{\hat{p}} = \hat{V}'_0(\hat{x}) - \hat{V}'_1(\hat{x}, \omega t) - \int_0^t dt' \gamma(t - t') \hat{p}(t') + \hat{\eta}(t) \tag{3}$$

In the present work, the expression for the damping kernel can be described as $\gamma(t - t') = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\Omega [J(\Omega)/\Omega] \cos \Omega(t - t')$ where the term $J(\Omega)$ stands for the spectral density of the harmonic bath and the term $\hat{\eta}(t)$ describes the noise characterized by,

$$\hat{\eta}(t) = \sum_{j=1}^{N} \left[m_j c_j {\Omega_j}^2 \{\hat{x}_j(0) - c_j \hat{x}(0)\} \cos \Omega_j t + \frac{\hat{p}_j(0)}{m_j \Omega_j} \sin \Omega_j t \right]$$

In the Ohmic regime, $(\Omega) = m\gamma\Omega$, where γ denotes the friction coefficient. Note that $\hat{\eta}(t)$ considered here is a zero-mean Gaussian random noise. The statistical properties of $\hat{\eta}(t)$ can be constructed by exploiting appropriate canonical thermal distribution of bath degrees of freedom at initial stage, $t = 0$,

$$\langle\hat{\eta}(t)\hat{\eta}(t') + \hat{\eta}(t')\hat{\eta}(t)\rangle_{\boldsymbol{QS}} = \hbar \int_{-\infty}^{+\infty} \frac{d\Omega}{\pi} J(\Omega) \coth\left(\frac{\hbar\boldsymbol{\Omega}}{2k_B T}\right) \cos\Omega(t - t') \tag{4}$$

Note that the above equation, Eq.(4), is the famous fluctuation-dissipation theorem/relation (FDT/FDR) which is a very useful instrument in chemical physics for monitoring the counter intuitive behavioural aspect of systems that follows structure of detailed balance. Here k_BT (k_B stands for Boltzmann constant) delineates the equilibrium thermal energy. In the model, the average is computed over the initial bath-variables as

$$\langle\hat{O}\rangle_{QS} = \frac{Tr[\hat{O}exp(-\hat{H}_B/k_B T)]}{Tr[exp(-\hat{H}_B/k_B T)]} \tag{5}$$

where $\langle\dots\rangle_{\boldsymbol{QS}}$ indicates a quantum statistical average over the bath-variables. As Eq. (3) is the generalized quantum mechanical operator form of Langevin equation (Shit, Chattopadhyay & Ray Chaudhuri, 2011b; Ghosh *et al.,* 2011), it is practically unmanageable and non-trivial task to obtain required solution. Therefore, it is practical to sketch Eq.(3) in an operator-free manner. Obeying the scheme suggested by Ray and co-workers (Barik & Ray, 2005), starting form a microscopic system-bath Hamiltonian, one gets the following *c*-number generalized quantum Langevin equation delineating the evolution of the expectation values position operators of the system under consideration:

$$m\dot{x} = p$$

$$\dot{p} = -U_0'(x) - U_1'(x, \omega t) - \gamma p + \eta(t) \tag{6}$$

Where

$$U_0' = V_0'\big(x(t)\big) - Q_V^0$$

$$U_1' = V_1'(x(t), \omega t) - Q_V^1 \tag{7}$$

Here

$$x(t) = \langle\hat{x}(t)\rangle_Q$$

$$p(t) = \langle\hat{p}(t)\rangle_Q$$

$$\eta(t) = \langle\hat{\eta}(t)\rangle_Q = \sum_j \left[m_j c_j \Omega_j^2 \left\{\langle\hat{x}_j(0)\rangle_Q - c_j\langle\hat{x}(0)\rangle_Q\right\} \cos\Omega_j t + \frac{\langle\hat{p}_j(0)\rangle_Q}{m_j\Omega_j} \sin\Omega_j t\right] \tag{8}$$

$\langle\dots\rangle_Q$ describes quantum mechanical average estimate (Barik, Banerjee & Ray, 2009). Here, Q_V^0 and Q_V^1 symbolize the quantum mechanical correction terms, described by

$$Q_V^0 = V_0'(x) - \langle V_0'(\hat{x})\rangle_Q$$

$$Q_V^1 = V_1'(x, \omega t) - \langle V_1'(\hat{x}, \omega t)\rangle_Q \tag{9}$$

The system parameters $\hat{x}$ and $\hat{p}$ can be described as

$$\hat{x}(t) = x(t) + \delta\hat{x}(t), \;\; \hat{p}(t) = p(t) + \delta\hat{p}(t), \tag{10}$$

Note that, in this model, the terms $x(=\langle\hat{x}\rangle_Q)$ and $p(=\langle\hat{p}\rangle_Q)$ can be considered as the quantum mechanical mean values. The operators $\delta\hat{x}$ and $\delta\hat{p}$ appear as quantum fluctuations around their respective average values and they obey:

$$\langle\delta\hat{x}(t)\rangle_Q = 0 = \langle\delta\hat{p}(t)\rangle_Q, \;\; [\delta\hat{x}, \delta\hat{p}] = i\hbar \tag{11}$$

Exploiting Eq. (10) along with the Taylor series based expansion around x (Bhattacharya, Chattopadhyay & Ray Chaudhuri, 2009),

$$Q_V^0 = -\sum_{n\geq 2}\frac{1}{n!}V_0^{n+1}(x)\langle\delta\hat{x}^n(t)\rangle_Q$$

$$Q_V^1 = -\sum_{n\geq 2}\frac{1}{n!}V_1^{n+1}(x,\omega t)\langle\delta\hat{x}^n(t)\rangle_Q \qquad (12)$$

Here the term $V^{n+1}(x)$ corresponds to the $(n+1)$th derivative of $V(x)$. Estimation of $Q_V^1(x,t),(i=0,1)$ relies on the quantum mechanical correction term $\langle\delta\hat{x}^n(t)\rangle_Q$ that can be computed exploiting the scheme mentioned in References (Shit, Chattopadhyay & Ray Chaudhuri, 2011b; Ghosh *et al.,* 2011). The following expression can be used to ascertain $\langle\hat{\eta}(t)\rangle_Q$ as a *c*-number noise:

$$\langle\langle\hat{\eta}(t)\rangle_Q\rangle_S = 0;$$

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = \frac{1}{2}\sum_{j=1}^{N} c_j^2\,\Omega_j^2\hbar\Omega_j\coth\left(\frac{\hbar\Omega_j}{2\mathrm{k}_BT}\right)\cos\Omega_j(t-t'); \qquad (13)$$

Eq.(13) clearly advocates that the noise $\langle\hat{\eta}(t)\rangle_Q$ fulfils the quantum FDR, and is emerged if and only if the initial mean-values of parameters (momenta and coordinates) of the bath oscillators have canonical thermal Wigner distribution, P_j for the displaced harmonic oscillator:

$$\mathrm{P}_j = N\exp\left\{-\frac{\langle\hat{p}_j(0)\rangle_Q^2+\Omega_j^2\left[\langle\hat{x}_j(0)\rangle_Q - c_j\langle\hat{q}(0)\rangle_Q\right]^2}{2\hbar\Omega_j\left(\bar{n}_j(\Omega_j)+\frac{1}{2}\right)}\right\} \qquad (14)$$

Here, *N* appears as normalization constant. Note that positive definite function, P_j depends on the initial preparation of the system under study. Structural properties of P_j remains applicable as a pure state, non-singular distribution even at *T* = 0. In the above expression $\bar{n}_j$ can be characterized as average photon number at temperature T:

$$\bar{n}_j = \left[\exp\left(\frac{\hbar\Omega_j}{2\mathrm{k}_BT}\right)-1\right]^{-1} \qquad (15)$$

In this model, the statistical average of any dynamical/observable O_j can be expressed as (quantum mechanical mean value)

$$\langle O_j(0)\rangle_S = \int O_j\,\mathrm{P}_j d\langle\hat{p}_j(0)\rangle d\{\langle\hat{x}_j(0)\rangle - \langle\hat{x}(0)\rangle\} \qquad (16)$$

As expected, for the case that $\hbar\omega \ll k_BT$ (thermal limit), the form of the distribution of quantum mechanical mean values of the bath oscillators converts to the corresponding form of the Maxwell–Boltzmann distribution, the classical one. It is to be mentioned here that Eqs. (8), (14) and (16) can be exploited to reveal the characteristic features of the *c*-number noise. Eq. (6) can be viewed as the desired *c*-number quantum Langevin equation (QLE). In this model, the two quantum correction terms Q_V^0 and Q_V^1 emerge from the nonlinear nature of the potential used. Using a suitable physically motivated approximation (Shit, Chattopadhyay & Ray Chaudhuri, 2011b; Ghosh, Shit, Chattopadhyay & Ray Chaudhuri, 2011), Eq. (13) can be described as where,

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = 2D_q\delta(t-t')$$

$$D_q = \left(\frac{\gamma\hbar\Omega_0}{2}\right)\coth\left(\frac{\hbar\Omega_j}{2\mathrm{k}_B T}\right) \quad (17)$$

Here Ω_0 corresponds to a common linearized system frequency for all bath modes. In the present development, the usual quantum statistical average is described by $\langle\langle \dots \rangle_Q\rangle_S$. It is to be noted here that although *c*-number noise $\eta(t)(=\langle\hat{\eta}(t)\rangle_Q)$ follows the FDR and furnishes the same anti-commutator just as operator noise term $\hat{\eta}(t)$, being a *c*-number term, the commutator form of the same vanishes. In that sense, the treatment present here is not fully quantum mechanical. Actually $\eta(t)$ is a classical-like noise in conjunction with quantum mechanical correction term. Thus, the present development can be considered as a semiclassical scheme. Basically, in the present work, the system is handled using a suitable quantum mechanical protocol, but the bath degrees of freedom have been managed semi-classically. It is worth mentioning that the complexity of treating the complicated operator quantum Langevin equation (OQLE) can be circumvented by exploiting the above mentioned semiclassical approach that attempts to handle the OQLE on the same footing as that of the classical LE while preserving the leading-order quantum effect. Note that in the high temperature quantum regime, $\frac{\hbar\Omega_0}{\mathrm{k}_B T} \ll 1$ (where the correction terms corresponding to the quantum effect emerge as a coupled infinite set of a hierarchy of equations) D_q can be approximated as $\gamma\mathrm{k}_B T$ and consequently, with this approximation, from Eq. (13), one can obtain the following form of the classical standard *δ*-correlated FDR, independent of the system frequency:

$$\langle\eta(t)\rangle_S = 0;$$
$$\langle\eta(t)\eta(t')\rangle_S = 2\gamma\mathrm{k}_B T\delta(t-t'), \quad (18)$$

Eq. (18) is the famous Einstein FDR in the Markovian limit. This scheme hence assists to get the classical form from the corresponding quantum mechanical relation.

Now, for the investigation of the dynamics of the present model, Eq. (6) with time dependent potential needs to be solved which is very tough to reach and generally can be achieved numerically by exploiting some physically motivated approx. scheme. In this model, the high frequency oscillating force exerts the force $\hat{F}(\hat{x},\omega t) = -\hat{V}_1'(\hat{x},\omega t)$. Here, the frequency ω is very large compared to all other pertinent system frequencies: $\omega\left(\omega > \frac{1}{\tau}\right)$. τ can be viewed as the order of magnitude of the period of motion of the system that it would perform in the field of $\hat{V}_0'(\hat{x})$. Therefore, the system does not have enough time to interact with the periodic force before the force alters sign. Under this condition, one can apply "Kapitsa-Landau time- window" in which the motion of the particle can be divided into a "slow" part as well as a "fast" one that comprises of a rapid motion around the "slow" part. It should be noted that the fast motion emerges in an effective potential for the "slow" motion. Therefore, focus on the following solution of Eq. (6):

$$x(t) = X(t) + \xi(X,\dot{X},\omega t) \quad (19)$$

Here, $X(t)$ describes the 'slow' part whereas ξ corresponds to the 'fast' part of the motion. In Ref. (Shit, Chattopadhyay & Ray Chaudhuri, 2012d), authors have demonstrated clearly that ξ will rely primarily on X and $\dot{X}$ rather on the higher order time derivatives. Here, ξ has been selected in such a fashion that Eq. (6) will yield a time-independent equation for X. Although it is very difficult to achieve exact solution exploiting Eq. (6) but in the limit of high frequencies, one can obtain in orders of $\frac{1}{\omega}$. Here, the following fast time variable is suggested

$$\tau = \omega t$$

for which the time average of ξ over one period disappears:

$$\bar{\xi} = \frac{1}{2\pi}\int_0^{2\pi} d\tau \xi\left(X, \dot{X}, \tau\right) = 0 \quad (20)$$

It is to be clarified here that ξ is not periodic in t instead, it should be a periodic function of the τ. In the present work, both times, τ and *t* are handled as independent parameters. In terms of 'fast time' variable, τ,

$$\frac{d\xi}{dt} = \omega\frac{d\xi}{d\tau} + \frac{d\xi}{dX}\dot{X} + \frac{d\xi}{d\dot{X}}\ddot{X} \quad (21)$$

and

$$\frac{d^2\xi}{dt^2} = \omega^2\frac{\partial\xi}{\partial\tau^2} + 2\omega\left[\frac{\partial^2\xi}{\partial X\partial\tau}\dot{X} + \frac{\partial^2\xi}{\partial\dot{X}\partial\tau}\ddot{X}\right] + \frac{\partial\xi}{\partial X}\ddot{X} + \frac{\partial\xi}{\partial\dot{X}}\dddot{X}$$
$$+\frac{\partial^2\xi}{\partial X^2}\dot{X}^2 + 2\frac{\partial^2\xi}{\partial X\partial\dot{X}}X\dddot{X} + \frac{\partial^2\xi}{\partial\dot{X}^2}\ddot{X}^2 \quad (22)$$

Exploiting Eqs. (21) and (22) along with Eq. (6) get the following form [for details, see reference (Shit, Chattopadhyay & Ray Chaudhuri, 2012a)]:

$$m\left\{\ddot{X} + \omega^2\frac{\partial\xi}{\partial\tau^2} + 2\omega\left(\frac{\partial^2\xi}{\partial X\partial\tau}\dot{X} + \frac{\partial^2\xi}{\partial\dot{X}\partial\tau}\ddot{X}\right) + \frac{\partial\xi}{\partial X}\ddot{X} + \frac{\partial\xi}{\partial\dot{X}}\dddot{X} + \frac{\partial^2\xi}{\partial X^2}\dot{X}^2 + 2\frac{\partial^2\xi}{\partial X\partial\dot{X}}\dot{X}\ddot{X} + \frac{\partial^2\xi}{\partial\dot{X}^2}\ddot{X}^2\right\} + \gamma\dot{X} +$$
$$\gamma\left\{\frac{\partial\xi}{\partial X}\dot{X} + \frac{\partial\xi}{\partial\dot{X}}\ddot{X} + \omega\frac{\partial\xi}{\partial\tau}\right\} = -U_0'(X+\xi) - U_1'(X+\xi,\tau) + \eta(t) \quad (23)$$

Here the slow dynamics usually controls the overall dynamics of the particle. Note that the initial noise term entirely has its impact on the slow dynamics, and it has no effect on the fast dynamics. At the high frequencies, ξ becomes be very small (of the order of $\frac{1}{\omega^2}$). Thus, one can expand $U_0(X+\xi)$ and $U_1(X+\xi,\tau)$ in powers of; U_0 and U_1 are assumed to be smooth functions of the coordinate. Therefore, one can expand ξ in powers of $\frac{1}{\omega}$:

$$\xi = \sum_{n=1}^{\infty}\frac{1}{\omega^n}\xi_n \quad (24)$$

Here, selection of ξ_i should be in such a fashion that the equation of *x* which emerges from Eq. (23) does not rely on time variable, τ. From Eq. (24) so,

$$m\left\{\ddot{X} + \omega^2\frac{\partial^2}{\partial\tau^2}\sum_n\frac{1}{\omega^n}\xi_n + 2\omega\left(\dot{X}\frac{\partial^2}{\partial X\partial\tau}\sum_n\frac{1}{\omega^n}\xi_n + \ddot{X}\frac{\partial^2}{\partial\dot{X}\partial\tau}\sum_n\frac{1}{\omega^n}\xi_n\right) + \ddot{X}\frac{\partial}{\partial X}\sum_n\frac{1}{\omega^n}\xi_n + \right.$$
$$\left.\dddot{X}\frac{\partial}{\partial\dot{X}}\sum_n\frac{1}{\omega^n}\xi_n + \dot{X}^2\frac{\partial^2}{\partial X^2}\sum_n\frac{1}{\omega^n}\xi_n + 2\dot{X}\ddot{X}\frac{\partial^2}{\partial X\partial\dot{X}}\sum_n\frac{1}{\omega^n}\xi_n + \ddot{X}^2\frac{\partial^2}{\partial\dot{X}^2}\sum_n\frac{1}{\omega^n}\xi_n\right\} + \gamma\dot{X} +$$
$$\gamma\left\{\dot{X}\frac{\partial}{\partial X}\sum_n\frac{1}{\omega^n}\xi_n + \ddot{X}\frac{\partial}{\partial\dot{X}}\sum_n\frac{1}{\omega^n}\xi_n + \omega\frac{\partial}{\partial\tau}\sum_n\frac{1}{\omega^n}\xi_n\right\} = -\left[\{U_0'(X)+U_1'(X,\tau)\} + \right.$$
$$\{U_0''(X)+U_1''(X,\tau)\} \times \sum_{n=1}^{\infty}\frac{1}{\omega^n}\xi_n + \frac{1}{2!}\{U_0'''(X)+U_1'''(X,\tau)\}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\omega^n}\frac{1}{\omega^m}\xi_n\xi_m +$$
$$\left.\cdots\right] + \eta(t)$$

(25)

To get the required equation, all the terms of the identical order are collected. From the very mode of discussion, it is evident that in the leading order of ω , the only contribution is

$$\frac{\partial^2\xi}{\partial\tau^2} = 0. \tag{26}$$

Therefore, without sacrificing generality, one may adopt:

$$\xi_1 = 0 \tag{27}$$

For ω^0 so

$$m\left\{\ddot{X} + \frac{\partial^2\xi_2}{\partial\tau^2}\right\} + \gamma\dot{X} = -U_0'(X) - U_1'(X,\tau) \tag{28}$$

To balance τ dependence, set

$$\frac{\partial^2\xi_2}{\partial\tau^2} = \frac{1}{m}U_1'(X,\tau) \tag{29}$$

ξ_2 must be periodic in time variable, τ. To circumvent secular terms, the time integral needs to have a vanishing average over a period.

$$\xi_2 = -\frac{1}{m}\int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau) \tag{30}$$

Substituting ξ_2 in Eq. (28), get

$$m\ddot{X} = -U_0'(X) - \gamma\dot{X}. \tag{31}$$

In the next order (ω^{-1}) get

$$m\left(\frac{\partial^2\xi_3}{\partial\tau^2} + 2\dot{X}\frac{\partial^2\xi_2}{\partial X\partial\tau} + 2\ddot{X}\frac{\partial^2\xi_2}{\partial\dot{X}\partial\tau}\right) + \gamma\frac{\partial\xi_2}{\partial\tau} = 0$$

From Eq. (30), it is evident that ξ_2 is not a function of $\dot{X}$ and hence the above equation reduces to

$$\frac{\partial^2\xi_3}{\partial\tau^2} = -2\dot{X}\frac{\partial^2\xi_2}{\partial X\partial\tau} - \gamma\frac{\partial\xi_2}{\partial\tau} \tag{32}$$

Now, using Eqs.(30) and (31), have the solution for ξ_3 as

$$\xi_3 = \frac{2}{m}\dot{X}\int_0^\tau d\tau\int_0^\tau d\tau\int_0^\tau d\tau\, U_1''(X,\tau) + \frac{\gamma}{m^2}\int_0^\tau d\tau\int_0^\tau d\tau\int_0^\tau d\tau\, U_1'(X,\tau) \tag{33}$$

Therefore, the terms of the order of ω^{-2} can be expressed as

$$m\left[\frac{\partial^2\xi_4}{\partial\tau^2} + 2\left(\dot{X}\frac{\partial^2\xi_3}{\partial X\partial\tau} + \ddot{X}\frac{\partial^2\xi_3}{\partial\dot{X}\partial\tau}\right) + \ddot{X}\frac{\partial\xi_2}{\partial X} + \dot{X}^2\frac{\partial^2\xi_2}{\partial X^2}\right] + \gamma\left[\dot{X}\frac{\partial\xi_2}{\partial X} + \frac{\partial\xi_3}{\partial\tau}\right] = -U_0''(X)\xi_2 - U_1''(X,\tau)\xi_2 \tag{34}$$

From Eq. (33), obtain the value of $\frac{\partial^2\xi_4}{\partial\tau^2}$ as

$$\frac{\partial^2\xi_4}{\partial\tau^2} = -\frac{U_0''(X)}{m^2}\int_0^\tau d\tau\int_0^\tau d\tau\, U_1'(X,\tau) + \frac{U_1''(X,\tau)}{m^2}\int_0^\tau d\tau\int_0^\tau d\tau\, U_1'(X,\tau) - \frac{3}{m}\dot{X}^2\int_0^\tau d\tau\int_0^\tau d\tau\, U_1'''(X,\tau) - \frac{3}{m}\ddot{X}\int_0^\tau d\tau\int_0^\tau d\tau\, U_1''(X,\tau) - \frac{3\gamma}{m^2}\dot{X}\int_0^\tau d\tau\int_0^\tau d\tau\, U_1''(X,\tau) - \frac{\gamma^2}{m^3}\int_0^\tau d\tau\int_0^\tau d\tau\, U_1'(X,\tau) \tag{35}$$

Let ξ_4 be periodic in τ. Now a function $f_1(X,\tau)$ is constructed as follows:

$$f_1(X,\tau) = \frac{1}{m^2} U_1''(X,\tau) \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau) - \frac{1}{m^2} \overline{U_1''(X,\tau) \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)}, \quad (36)$$

so that $\bar{f}_1(X,\tau) = 0.$

Now choose ξ_4 as

$$\xi_4 = \frac{U_0''(X)}{m^2} \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau) + \int_0^\tau d\tau \int_0^\tau d\tau\, f_1(X,\tau) -$$
$$\frac{3}{m} \dot{X}^2 \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'''(X,\tau) - \frac{3}{m} \ddot{X} \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau) -$$
$$\frac{3\gamma}{m^2} \dot{X} \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau) - \frac{\gamma^2}{m^3} \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau) \quad (37)$$

This solution equivalences all the τ-dependence of Eq.(35) and yields the following extra term,

$$\frac{1}{m^2} \overline{U_1''(X,\tau) \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)}$$

which for slow dynamics yields,

$$m\ddot{X} + \gamma \dot{X} = -U_0'(X) + \frac{1}{m\omega^2} \overline{U_1''(X,\tau) \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + O(\omega^{-3}) \quad (38)$$

The above equation can be considered as the leading order correction corresponding to the periodic potential U_1. The terms of the order of ω^{-3} of Eq. (25) provide:

$$m\left[\frac{\partial^2 \xi_5}{\partial \tau^2} + 2\left(\dot{X}\frac{\partial^2 \xi_4}{\partial X \partial \tau} + \ddot{X}\frac{\partial^2 \xi_4}{\partial \dot{X} \partial \tau}\right) + \ddot{X}\frac{\partial \xi_3}{\partial X} + \dddot{X}\frac{\partial \xi_3}{\partial \dot{X}} + \dot{X}^2 \frac{\partial^2 \xi_3}{\partial X^2} + 2\dot{X}\ddot{X}\frac{\partial \xi_3}{\partial X \partial \dot{X}} + \dot{X}^2 \frac{\partial^2 \xi_3}{\partial \dot{X}^2}\right] +$$
$$\gamma\left[\dot{X}\frac{\partial \xi_3}{\partial X} + \ddot{X}\frac{\partial \xi_3}{\partial \dot{X}} + \frac{\partial \xi_4}{\partial \tau}\right] = -U_0''(X)\xi_3 - U_1''(X,\tau)\xi_3 \quad (39)$$

Now, ξ_5 will be chosen in a manner ensuring that it shall scarp all the periodic terms with vanishing average. Note that entire terms of LHS of Eq. (39) has a vanishing average. Consequently, only the terms of RHS of Eq. (39) shall contribute to the slow coordinate. This contribution will result only from $\overline{-U_0''}\,\xi_3 - \overline{U_1''\xi_3}$ and, as the term $\overline{U_0''\xi_3}$ vanishes as per this model, one may easily obtain,

$$\overline{-U_1''\xi_3} = \frac{\gamma}{m^2} \overline{\int_0^\tau d\tau\, U_1''(X,\tau) \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} \quad (40)$$

Incorporating the effect of this term [i.e., $O(\omega^{-3})$] into the slow dynamics, one will have (using integration by parts),

$$m\ddot{X} + \gamma \dot{X} = -U_0'(X) + \frac{1}{m\omega^2} \overline{\int_0^\tau d\tau\, U_1'(X,\tau) \int_0^\tau d\tau\, U_1''(X,\tau)} +$$
$$\frac{\gamma}{m^2\omega^3} \overline{\int_0^\tau d\tau\, U_1''(X,\tau) \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + O(\omega^{-4}) \quad (41)$$

the contribution of terms $O(\omega^{-4})$ to the equation of X is

$$\overline{-U_0''\xi_4} - \overline{U_1''\xi_4} - \frac{1}{2}\overline{U_0'''\xi_2^2} - \frac{1}{2}\overline{U_1'''\xi_2^2} \quad (42)$$

Although the averages of other terms have finite values, but the first term will vanish again. From Eqs.(30) and (37), one gets

$$\overline{-U_0''(X)\xi_4} - \overline{U_1''(X,\tau)\xi_4} - \frac{1}{2}\overline{U_0'''(X)\xi_2^2} - \frac{1}{2}\overline{U_1'''(X,\tau)\xi_2^2} = \frac{1}{2m^2}U_0'''(X)\overline{\left[\int_0^\tau d\tau\ \int_0^\tau d\tau\, U_1'(X,\tau)\right]^2} - \frac{1}{2m^2}U_1'''(X,\tau)\overline{\left[\int_0^\tau d\tau\ \int_0^\tau d\tau\, U_1'(X,\tau)\right]^2} + \frac{U_0''(X)}{m^2}\overline{\int_0^\tau d\tau U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + \frac{1}{m^2}\overline{\int_0^\tau d\tau U_1''(X,\tau)\ \int_0^\tau d\tau\, \left[U_1'(X,\tau)\int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)\right]} - \frac{3}{m}\dot{X}^2\overline{\int_0^\tau d\tau U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'''(X,\tau)} - \frac{3}{m}\ddot{X}\overline{\int_0^\tau d\tau U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau)} - \frac{3\gamma}{m^2}\dot{X}\overline{\int_0^\tau d\tau U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau)} - \frac{\gamma^2}{m^3}\overline{\int_0^\tau d\tau\, U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)}.$$

Including these terms $O(\omega^{-4})$ into the slow dynamic provide the EOM for slow variable as

$$m\ddot{X} + \gamma\dot{X} = -U_0'(X) - \frac{1}{m\omega^2}\overline{\int_0^\tau d\tau\, U_1'(X,\tau)\int_0^\tau d\tau\, U_1''(X,\tau)} + \frac{\gamma}{m^2\omega^3}\overline{\int_0^\tau d\tau\, U_1''(X,\tau)\int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} - \frac{1}{2m^2\omega^4}U_0'''(X)\overline{\left[\int_0^\tau d\tau\ \int_0^\tau d\tau\, U_1'(X,\tau)\right]^2} - \frac{1}{2m^2\omega^4}U_1'''(X,\tau)\overline{\left[\int_0^\tau d\tau\ \int_0^\tau d\tau\, U_1'(X,\tau)\right]^2} - \frac{1}{m^2\omega^4}U_0''(X)\overline{\int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} - \frac{1}{m^2\omega^4}\overline{U_1''(X,\tau)\int_0^\tau d\tau\ \int_0^\tau d\tau\, U_1''(X,\tau)\int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + \frac{3}{m\omega^4}\dot{X}^2\overline{\int_0^\tau d\tau \int_0^\tau d\tau U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'''(X,\tau)} + \frac{3}{m\omega^4}\ddot{X}\overline{\left[\int_0^\tau d\tau\ \int_0^\tau d\tau\, U_1''(X,\tau)\right]^2} + \frac{3\gamma}{m^2\omega^4}\dot{X}\overline{\left[\int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau)\right]^2} + \frac{\gamma^2}{m^3\omega^4}\overline{\int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + O(\omega^{-4}) + \eta(t) \tag{43}$$

Substitution of $\ddot{X}$ by $-\frac{V_0'}{m} - \frac{\gamma}{m}\dot{X}$, incorporates a very insignificant error of the order of $\omega^{-4}[O(\omega^{-4})]$ in Eq. (43). Performing suitable manipulation of algebra, finally get the following the EOM for the slow part (corrected up to the order ω^{-4})

$$m\ddot{X} + \gamma\dot{X} = -U_{eff}'(X) + \frac{3}{m\omega^4}\dot{X}^2\overline{\int_0^\tau d\tau \int_0^\tau d\tau U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'''(X,\tau)} + \frac{3U_0'}{m^2\omega^4}\overline{\left[\int_0^\tau d\tau\ \int_0^\tau d\tau\, U_1''(X,\tau)\right]^2} + \frac{\gamma}{m^2\omega^3}\overline{\int_0^\tau d\tau\, U_1''(X,\tau)\int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + \frac{\gamma^2}{m^3\omega^4}\overline{\int_0^\tau d\tau \int_0^\tau d\tau\, U_1''(X,\tau)\ \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + \eta(t) \tag{44}$$

where $U_{eff}(X)$ can be considered as the effective potential and can described as,

$$U_{eff}(X) = U_0(X) - \frac{1}{2m\omega^2}\overline{\left[\int_0^\tau d\tau\, U_1'(X,\tau)\right]^2} + \frac{1}{2m^2\omega^4}\overline{U_1''(X,\tau)\left[\int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)\right]^2} + \frac{1}{2m^2\omega^4}\overline{U_0''(X)\left[\int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)\right]^2} \tag{45}$$

It is to be noted here that the noise term only arises in the EOM of the slow variable, and the γ-containing terms are explicitly dropped from arising in the equation of U_{eff}, rather, two other terms of EOM contain it owing to the dissipative surrounding. If there is no interaction of the system with surrounding, the γ-containing terms do not arise in the dynamical equation. Here quantum effect is manifested in Eq. (44) in U_0 and U_1 through the correction terms Q_V^0 and Q_V^1 . If one had used the classical calculation only, there would be no contribution of Q_V^0 and Q_V^1. Then, the contribution of U_0 and U_1 would have been replaced by V_0 and V_1. In the classical limit ($\hbar \to 0$), on incorporation of terms O (ω^{-3}), Eq.(44) converts to the following form

$$m\ddot{X} + \gamma\dot{X} = -U_0'(X) + \frac{1}{m\omega^2}\overline{\int_0^\tau d\tau U_1'(X,\tau) \int_0^\tau d\tau\, U_1''(X,\tau)} + \frac{\gamma}{m^2\omega^3}\overline{\int_0^\tau d\tau\, U_1''(X,\tau) \int_0^\tau d\tau \int_0^\tau d\tau\, U_1'(X,\tau)} + \eta(t)$$

which is identical with equation (28) of Ray Chaudhury and co-workers (Shit, Chattopadhyay & Ray Chaudhuri, 2012a). At this point, it is important to note that in the previous works (Shit, Chattopadhyay & Ray Chaudhuri, 2012, Shit, Chattopadhyay & Ray Chaudhuri, 2011a), pure quantum mechanical model upto the order ω^{-2} has been published. From the expression of effective time independent Langevin equation [Eq. (44)] of this description, different dynamical studies may be performed. By defining a particular model system potential, Eq. (44) may be numerically simulated (Shit, 2016) to investigate the escape rate of the perturbed/dressed particle and the influence of the external modulation on the resulting rate along with the thermal noise may also be studied. The effective potential described and characterized above has the capability to confine the particle. It is very crucial to note that a bound rapidly oscillating potential confines systems even if its time average disappears. This work may also have some applications. For instance, depending on the nature of spatial variation of the applied forces, the effective potential defined above often have more than one local minimum even if the initial (unmodified) potential has only single local minima. Under such situations, a collection of Brownian particles would tend to segregate in two separate collections.

Conclusion:

Reaction and feedback of a dynamical system to a rapidly oscillating periodic driving force are two of the most challenging and fundamental issues in the realm of chemical dynamics in condensed phases. Time dependent rapidly driven quantum dissipative systems exhibit an intricate interplay of linearity, SB coupling, and nonequilibrium behaviour as a result of the time dependent driving. In this work, beginning from a quantum mechanical system-bath Hamiltonian with explicit time-dependence, using multiple scale perturbation theory, an effective time-independent *c*-number generalized Langevin equation at leading order is derived with an effective time independent potential which permits one to survey the dynamics of the system under the influence of rapidly oscillating fields, in the architecture of methodologies that were designed for systems in the presence of time-independent potentials. Calculations for the dynamics of the slow part have been done perturbatively in powers of the frequency (ω) of the external driving force used to the order of $1/\omega^4$. The described work may contribute to the microscopic understanding of barrier crossing phenomena under the impact of a rapidly oscillating field in condensed phases, and in particular non-adiabatic effects and may help to understand and interpret many experimental results.

Acknowledgement

The author expresses her gratitude to Professor Sudip Kumar Chattopadhyay and Professor Jyotipratim Ray Chaudhuri, and is thankful to Kandi Raj College, Kandi, Murshidabad, West Bengal, India.

References

Barik, D., & Ray, D. S. (2005). Quantum state-dependent diffusion and multiplicative noise: a microscopic approach. *Journal of Statistical Physics*, *120*, 339-365. https://doi.org/10.1007/s10955-005-5251-y

Barik, D., Banerjee, D., & Ray, D. S. (2009). *Quantum Brownian motion in C-numbers: Theory and applications*. Nova Science Publishers.

Bhattacharya, S., Chattopadhyay, S., & Ray Chaudhuri, J. (2009). Investigation of Noise-Induced Escape Rate: A Quantum Mechanical Approach. *Journal of Statistical Physics*, *136*, 733-750. https://doi.org/10.1007/s10955-009-9802-5

Bukov, M., D'Alessio, L., & Polkovnikov, A. (2015). Universal high-frequency behavior of periodically driven systems: from dynamical stabilization to Floquet engineering. *Advances in Physics*, *64*(2), 139-226. https://doi.org/10.1080/00018732.2015.1055918

Chen, J. C., Hambro, L., Sinfailam, A. L., & Chung, K. T. (1973). Multiple-Scattering Expansions for Nonrelativistic Three-Body Collision Problems. VIII. Eikonal Approximation for the Faddeev-Watson Multiple-Scattering Expansion. *Physical Review A*, *7*(6), 2003. https://doi.org/10.1103/PhysRevA.7.2003

Denisov, S. I., Polyakov, A. Y., & Lyutyy, T. V. (2011). Resonant suppression of thermal stability of the nanoparticle magnetization by a rotating magnetic field. *Physical Review B*, *84*(17), 174410. https://doi.org/10.1103/PhysRevB.84.174410

Denisov, S. I., Sakmann, K., Talkner, P., & Hänggi, P. (2006). Mean first-passage times for an ac-driven magnetic moment of a nanoparticle. *Europhysics Letters*, *76*(6), 1001. https://doi.org/10.1209/epl/i2006-10403-3

Denisov, S. I., Sakmann, K., Talkner, P., & Hänggi, P. (2007). Rapidly driven nanoparticles: Mean first-passage times and relaxation of the magnetic moment. *Physical Review B*, *75*(18), 184432. https://doi.org/10.1103/PhysRevB.75.184432

Floquet, G. (1883). Sur les équations différentielles linéaires à coefficients périodiques. In *Annales Scientifiques de l'École Normale Supérieure, 12*, 47-88. https://doi.org/10.24033/asens.220m

Ghosh, P., Shit, A., Chattopadhyay, S., & Chaudhuri, J. R. (2011). A semiclassical approach to explore the bistable kinetics of a Brownian particle in a nonequilibrium environment. *Journal of Statistical Mechanics: Theory and Experiment*, *2011*(02), P02026. https://doi.org/10.1088/1742-5468/2011/02/P02026

Hänggi, P. (2002). Stochastic resonance in biology how noise can enhance detection of weak signals and help improve biological information processing. *ChemPhysChem*, *3*(3), 285-290. https://doi.org/10.1103/RevModPhys.70.223

Hänggi, P., Łuczka, J., & Spiechowicz, J. (2020). Many faces of nonequilibrium: anomalous transport phenomena in driven periodic systems. *arXiv preprint arXiv:2002.11690*. https://doi.org/10.5506/APhysPolB.51.1131

Jung, P. (1993). Periodically driven stochastic systems. *Physics Reports*, *234*(4-5), 175-295. https://doi.org/10.1016/0370-1573(93)90022-6

Kapitsa, P. L. (1951). THERMAL CONDUCTANCE AND DIFFUSTION IN LIQUID MEDIUM IN THE CASE OF PERIODIC FLOW. I. DETERMINATION OF COEFFICIENT OF WAVE TRANSFER IN A PIPE, SLIT, AND CHANNEL. *Zhur. Eksptl'i Teoret. Fiz.*, *21*.

Kim, C., Talkner, P., Lee, E. K., & Hänggi, P. (2010). Rate description of Fokker–Planck processes with time-periodic parameters. *Chemical Physics*, *370*(1-3), 277-289. https://doi.org/10.1016/j.chemphys.2009.10.027

Landau, L. D., & Lifshitz, E. M. (2013). *Quantum Mechanics: Non-Relativistic Theory* (Vol. 3). Elsevier.

Paul, W. (1990). Electromagnetic traps for charged and neutral particles. *Reviews of Modern Physics*, *62*(3), 531. https://doi.org/10.1103/RevModPhys.62.531

Shirley, J. H. (1965). Solution of the Schrödinger equation with a Hamiltonian periodic in time. *Physical Review*, *138*(4B), B979. https://doi.org/10.1103/PhysRev.138.B979

Shit, A. (2016). High-frequency behavior of periodically modulated stochastic quantum system: activated escape. *Journal of Statistical Mechanics: Theory and Experiment*, *2016*(5), 053204. https://doi.org/10.1088/1742-5468/2016/05/053204

Shit, A., Chattopadhyay, S., & Chaudhuri, J. R. (2011a). Effective quantum Brownian dynamics in presence of a rapidly oscillating space-dependent time-periodic field. *Physical Review E*, *83*(6), 060101. https://doi.org/10.1103/PhysRevE.83.060101

Shit, A., Chattopadhyay, S., & Chaudhuri, J. R. (2011b). Towards an understanding of escape rate and state dependent diffusion for a quantum dissipative system. *Chemical Physics*, *386*(1-3), 56-72. https://doi.org/10.1016/j.chemphys.2011.06.002

Shit, A., Chattopadhyay, S., & Chaudhuri, J. R. (2012a). Controlling activated processes of nonadiabatically, periodically driven dynamical systems: A multiple scale perturbation approach. *The Journal of Chemical Physics*, *136*(23), 234506. https://doi.org/10.1063/1.4729848

Shit, A., Chattopadhyay, S., & Chaudhuri, J. R. (2012b). Escape rate for a quantum particle moving in a time-periodic rapidly oscillating potential: A time-independent approach. *Physical Review E*, *85*(5), 051102. https://doi.org/10.1103/PhysRevE.85.051102

Shit, A., Chattopadhyay, S., & Chaudhuri, J. R. (2012c). Quantum escape in the presence of a time-periodic oscillating force. *Europhysics Letters*, *97*(4), 40006. https://doi.org/10.1209/0295-5075/97/40006

Shit, A., Chattopadhyay, S., & Chaudhuri, J. R. (2013). Kapitza-Landau time window for a periodically driven system with friction: a system-bath Hamiltonian approach. *The European Physical Journal B*, *86*, 1-9. https://doi.org/10.1140/epjb/e2012-30852-0

Shit, A., Chattopadhyay, S., & Ray Chaudhuri, J. (2012d). Time-independent description of rapidly driven systems in the presence of friction: Multiple scale perturbation approach. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *22*(1), 013131. https://doi.org/10.1063/1.3692970

Spiechowicz, J., Hänggi, P., & Łuczka, J. (2022). Velocity multistability vs. ergodicity breaking in a biased periodic potential. *Entropy*, *24*(1), 98. https://doi.org/10.3390/e24010098

Spiechowicz, J., Marchenko, I. G., Hänggi, P., & Łuczka, J. (2023). Diffusion coefficient of a Brownian particle in equilibrium and nonequilibrium: Einstein model and beyond. *Entropy*, *25*(1), 42. https://doi.org/10.3390/e25010042

Weiss, U. (2012). *Quantum dissipative systems*.

Zwanzig, R. (1961). Lectures in Theoretical Physics vol 3 ed WE Brittin. *BW Downs and J Downs (New York: Interscience)*.

Zwanzig, R. (1973). Nonlinear generalized Langevin equations. *Journal of Statistical Physics*, *9*(3), 215-220. https://doi.org/10.1007/BF01008729