Doi: 10.60072/book.icoge2026.116

Boundary Value Problem for the Holmgren Equation with a Singular Coefficient in the Right Half-plane

Sarsenbaeva Gozzal[1*], Ruziev M.Kh[2]

[1] Karakalpak State University, Nukus, Uzbekistan
[2] V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan

[*]***Corresponding author:*** *gozzalsarsenbaeva13@gmail.com*

Abstract
In this paper, a boundary value problem for an elliptic equation degenerating within the domain is studied. By applying the Hankel transform and the method of separation of variables, the solution to the Dirichlet problem is presented in explicit form. The problem under study is equivalently reduced to solving a singular integral equation with a Cauchy kernel. Using the Carleman-Vekua method, its solution is found in explicit form.

Keywords: Bessel Function, Carleman-Vekua Method, Index of the Equation, Singular Integral Equation, Vertical Strip

Introduction

According to the Singular partial differential equations, appear during the study of several of aerodynamics and gas dynamics and irrigation problems. The Dirichlet problem and a Neumann-type problem for an elliptic equation in a vertical semi-strip were studied in. A Neumann-type boundary value problem for the Holmgren equation with a singular coefficient and a spectral parameter was solved in. A nonlocal problem for a class of equations of mixed type was investigated in.

Methodology

Consider the equation

$$|y|^m u_{xx} + u_{yy} - \frac{m}{2y} u_y = 0, m > 0 \tag{1}$$

in the domain $D = D^+ \cup D^- \cup I$

where $D^+ = \{(x, y) : x > 0, y > 0\}$

$D^- = \{(x, y) : x > 0, y < 0\}$

$I = \{(x, y) : x > 0, y = 0\}$

In the domain D we study the following boundary value problem for equation.

Problem D. Find a function $u = u(x, y)$ in the domain D satisfies the following conditions:

1. $u(x,y) \in C(\bar{D}) \cap C^2(D^+ \cup D^-)$ and satisfies equation (1) in $D^+ \cup D^-$;

2. $$\lim_{R\to\infty} u(x,y) = 0, \quad R^2 = x^2 + \frac{4}{(m+2)^2} |y|^{m+2}$$
(2)
$x > 0, y \in (-\infty, \infty)$;

3. $u = u(x,y)$ satisfies boundary conditions
$$u(0,y) = \varphi(y), y \in (-\infty, +\infty) \qquad (3)$$
and the transmisson condition
$$\lim_{y\to+0} y^{\beta_0} u_y = \lim_{y\to-0} (-y)^{\beta_0} u_y, \quad x \in (0,\infty) \qquad (4)$$
where these limits as $x = 0, x = \infty$ may have singularities of order less than one, where
$$\varphi(y) = \begin{cases} \varphi^+(y), & y \in (0,\infty), \\ \varphi^-(y), & y \in (-\infty, 0), \\ 0, & y = 0, \end{cases}$$
and
$$\varphi^+(y) \in C[0,\infty), \quad \varphi^-(y) \in C(-\infty, 0],$$
$$\varphi^+(0) = \varphi^-(0) = 0, \quad \varphi(\infty) = 0.$$
In this work, we will apply the method of integral equations.

Findings and Discussion

Theorem. Suppose that conditions
$$y^{\frac{m}{2}} \varphi^+(y) \in L(0,\infty), \quad (-y)^{\frac{m}{2}} \varphi^-(y) \in L(-\infty, 0)$$
are satisfied. Then, problem D is uniquely solvable.

Conclusion

In this paper, the Dirichlet problem for the Holmgren equation with a singular coefficient in a half-plane is studied. The proof is carried out using the method of integral equations. The problem is reduced to solving a singular integral equation, and its solution is obtained in closed form via the Carleman-Vekua method.

References

Bers L. *Mathematical aspects of subsonic and transonic gas dynamics*. 1958, New York: Wiley.

N. I. Muskhelishvili. "*Singular Integral Equations: Boundary Problems of Function Theory and some applications of them to Mathematical Physics*" Moscow, 1962, 599 p.

Polubarinova-Kochina P. Ya., Pryazhinskaya V. G., Emikh V. N. *Matematicheskiye metody v voprosakh orosheniya.* 1969, Moscow: Nauka.

Ruziev M. Kh. *A boundary value problem for the Holmgren equation with singular coefficient and spectral parameter*. Journal of Elliptic and Parabolic equations. (2019) 5: pp. 269–280.

Ruziev M., Kazakbaeva K. *A nonlocal problem for a class of mixed-type equations in a domain with elliptic part as a half-strip*. Bulletin of the Institute of Mathematics, 2025, vol. 8, No. 2, pp. 66–71.

Serbina L. I. *A problem for the linearized Boussinesq equation with a nonlocal Samarskii condition*. Differential equations. 2002, vol. 38, Issue 8, pp. 1187–1194.

Shimkovich E. V. *On weighted boundary value problems for a degenerate equation of elliptic type in a half-strip.* Lithuanian Mathematical Journal. 1990, vol. 30, Issue 1, pp. 185–196.